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A Study of Theorems Involving The Laplace transform And Aleph ($\aleph$) -Function With Application

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Abstract: In this paper, the author establishes four interesting theorems exhibiting interconnections between images and originals of related functions in the Laplace transform. Further, we obtain five new and general integrals by the application of the theorems. Two known results are also given as a direct consequence of the third theorem. The importance of our findings lies in the fact that they involve the $\aleph$ -function which are very general in nature and are capable of yielding a large number of simpler and useful integrals merely by specializing the parameters in them.

Keywords: $\aleph$ -function, Laplace transform, Goldstein theorem.
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I. INTRODUCTION

The Laplace transform occurring in the paper will be defined in the following usual manner:

$$\overline{f}(s)=L\{f(x);s\}=\int_0^\infty e^{-sx}f(x)dx \tag{1.1}$$

Where Re(s)>0 and the function $f(x)$ is such that the integral on the R.H.S. of (1.1) is absolutely convergent.

The well known Parseval Goldstein theorem for the transform will be in the sequel:

If $\overline{f}_1(s)=L\{f_1(x);s\}$ and $\overline{f}_1(s)=L\{f_1(x);s\}$

Then $\int_0^\infty f_1(x)\,\overline{f_2}(x)\,dx=\int_0^\infty f_2(x)\,\overline{f_1}(x)\,dx$ (1.2)

The $\aleph$ - function introduced by Suland et.al. [6] defined and represented in the following form:

$$\aleph[z]=\aleph^{m,n}_{p_i,q_i;\tau_i;r}[z]=\aleph^{m,n}_{p_i,qi;\tau_i;r}\left[z\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_L\theta(s)z^s ds \tag{1.3}$$

Where $\omega=\sqrt{-1}$;

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s\right\}} \tag{1.4}$$

We shall use the following notation:

$A^*=\left(a_j,\alpha_j\right)_{1,n},[\tau_i\left(a_{ji},\alpha_{ji}\right)]_{n+1,p_i}$, $B^*=\left(b_j,\beta_j\right)_{1,m},[\tau_i\left(b_{ji},\beta_{ji}\right)]_{m+1,q_i}$

The following Laplace transforms will be required to prove our theorems.

$$s^{-\rho}\aleph^{m,n}_{p_i,q_i:\tau_i:r}[zs^{-\lambda}\left|^{A*}_{B*}\right.] = L\left\{s^{\rho-1}\aleph^{m,n}_{p_i,q_i+1:\tau_i:r}[zx^{\lambda}\left|^{A*}_{B*,(\rho,\lambda)}\right.]; s\right\} \quad (1.5)$$

Where $\min\left\{\min_{1\le j\le m}\operatorname{Re}\left(\rho+\lambda\tau_i\frac{b_{ji}}{\beta_{ji}}\right), \operatorname{Re}(s), \lambda\right\} > 0.$

$$s^{-\rho}\aleph^{m,n}_{p_i,q_i:\tau_i:r}[zs^{-\lambda}\left|^{A*}_{B*}\right.] = L\left\{s^{\rho-1}\aleph^{m,n}_{p_i+1,q_i:\tau_i:r}[zx^{\lambda}\left|^{A*,(1-\rho,\lambda)}_{B*}\right.]; s\right\} \quad (1.6)$$

Where $\max_{1\le j\le m}\operatorname{Re}\left(\lambda\tau_i\frac{a_{ji}}{\alpha_{ji}}-\rho\right) < 0, \{\operatorname{Re}(s), \lambda\} > 0$

II. THE THEOREMS:

Theorem 2.1:

If $L\{f(x); s\} = \overline{f}(s)$ (2.1)

And

$$L\left\{x^{\rho-1}\overline{f}(x)\aleph^{m,n}_{p_i,q_i+1:\tau_i:r}[zx^{\lambda}\left|^{A*}_{B*,(1-\rho,\lambda)}\right.]; s\right\} = h(s) \quad (2.2)$$

Then

$$\int_0^\infty (x+s)^{-\rho} f(x)\aleph^{m,n}_{p_i,q_i:\tau_i:r}[zx^{\lambda}\left|^{A*}_{B*}\right.]dx = h(s) \quad (2.3)$$

Where $\min_{1\le j\le m}\operatorname{Re}\left(\lambda\tau_i\frac{b_{ji}}{\beta_{ji}}+\rho\right) > 0$, $\min\{\operatorname{Re}(s), \lambda\} > 0$ and the integrals involved in equations (2.1), (2.2) and (2.3) are absolutely convergent.

Theorem 2.2:

If $L\{f(x); s\} = \overline{f}(s)$ (2.4)

And

$$L\left\{x^{\rho-1}e^{-ax}\,\overline{f}(x)\aleph^{m,n}_{p_i,q_i+1:\tau_i:r}[zx^{\lambda}\left|^{A*}_{B*,(1-\rho,\lambda)}\right.]; s\right\} = h(s) \quad (2.5)$$

Then

$$\int_0^\infty (x+s)^{-\rho} f(x-a)\aleph^{m,n}_{p_i,q_i:\tau_i:r}[z(x+s)^{-\lambda}\left|^{A*}_{B*}\right.]dx = h(s) \quad (2.6)$$

Where $\min_{1\le j\le n}\operatorname{Re}\left(\lambda\tau_i\frac{1-a_{ji}}{\beta_{ji}}+\rho\right) > 0$, $\min\{\operatorname{Re}(s), \lambda\} > 0, a \ge 0$ and the integrals involved are absolutely convergent.

Theorem 2.3:

If $L\{f(x); s\} = \overline{f}(s)$ (2.7)

And

$$L\left\{x^{\rho-1}e^{-ax}\,\overline{f}(x)\aleph^{m,n}_{p_i+1,q_i:\tau_i:r}[zx^{\lambda}\left|^{A*,(1-\rho,\lambda)}_{B*}\right.]; s\right\} = h(s) \quad (2.8)$$

Then

$$\int_0^\infty (x+s)^{-\rho} f(x)\aleph^{m,n}_{p_i,q_i:\tau_i:r}[z(x+s)^{\lambda}\left|^{A*}_{B*}\right.]dx = h(s) \quad (2.9)$$

Where $\max_{1\le j\le m}\operatorname{Re}\left(\lambda\tau_i\frac{b_{ji}}{\beta_{ji}}-\rho\right) < 0$, $\min\{\operatorname{Re}(s), \lambda\} > 0, a \ge 0$ and the integrals involved are absolutely convergent.

Theorem 2.4:

If $L\left\{ f(x);s \right\} = \overline{f}(s)$ (2.10)

And

$$L\left\{ x^{-\rho}\, \overline{f}(x) \aleph^{m,n}_{p_i,q_i;\tau_i;r}[zx^{\lambda} \left|^{A*}_{B*}\right.]; s \right\} = h(s) \qquad (2.11)$$

Then

$$\int_0^{\infty} (x+s)^{\rho-1} \overline{f}(x) \aleph^{m,n}_{p_i,q_i+1;\tau_i;r}[zx^{\lambda} \left|^{A*}_{B*,(\rho,\lambda)}\right.] dx = h(s) \qquad (2.12)$$

Where $\max_{1\le j\le m} \operatorname{Re}\left(\lambda \tau_i \frac{b_{ji}}{\beta_{ji}} - \rho \right) < 0$, $\min\{\operatorname{Re}(s), \lambda\} > 0, b \ge 0$ and the integrals involved are absolutely convergent.

III. INTEGRALS:

By specializing f(x), in the above theorem/ corollaries we can obtain new integrals involving $\aleph$ -functions. Thus, in Theorem 2.1, if we take $f(x) = (x^2 + 2ax)^{\nu-1/2}$,

The following integral follows after a little simplification with the help of ([5], p.138, eq. (13)):

$$\int_0^{\infty} (x^2+2ax)^{\nu-1/2} (x+s)^{-\rho} \aleph^{m,n}_{p_i,q_i;\tau_i;r}[z(x+s)^{-\lambda} \left|^{A*}_{B*}\right.] dx$$

$$= \frac{\sqrt{\pi}}{2\sin \nu\pi} \Gamma(\nu+1/2)(2a)^{r} [\frac{1}{(s-a)^{\rho-2\nu}} \sum_{r=0}^{\infty} \frac{(a/2)^{\nu+2r}}{r!\Gamma(-\nu+r+1)(s-a)^{2r}}$$

$$\aleph^{m,n+1}_{p_i+1,q_i+1;\tau_i;r}[z(s-a)^{-\lambda} \left|^{(\rho-2\nu+2r,\lambda),A*}_{B*,(\rho,\lambda)}\right.]$$

$$- \frac{1}{(s-a)^{\rho}} \sum_{r=0}^{\infty} \frac{(a/2)^{\nu+2r}}{r!\Gamma(-\nu+r+1)(s-a)^{2r}} \aleph^{m,n+1}_{p_i+1,q_i+1;\tau_i;r}[z(s-a)^{-\lambda} \left|^{(\rho-2\nu+2r,\lambda),A*}_{B*,(\rho,\lambda)}\right.] \;] \qquad (3.1)$$

Provided $\nu > -1/2$ and $|\arg(a)| < \pi$, $\min\left\{ \min_{1\le j\le m} \operatorname{Re}\left(\rho + \lambda\tau_i \frac{b_{ji}}{\beta_{ji}} \right), \operatorname{Re}(s), \lambda \right\} > 0.$

If we reduce the $\aleph$ -functions involved in (3.1) to $\aleph$ -function, we get the result in a very elegant form, after a little simplification:

$$\int_0^{\infty} (x^2+2ax)^{\nu-1/2} (x+s)^{-\rho} \aleph^{m,n}_{p_i,q_i;\tau_i;r}[z(x+s)^{-\lambda} \left|^{A*}_{B*}\right.] dx$$

$$= \frac{\sqrt{\pi}}{2\sin \nu\pi} \frac{\Gamma(\nu+1/2)(2a)^{r}}{(s-a)^{\rho-r}} \left(\frac{a}{2(s-a)} \right)^{-\nu}$$

$$[\aleph^{0,1;m,n;1,0}_{1,0;p_i,q_i+1;0,2;\tau_i;r}[{}^{z(s-a)^{-\lambda}}_{-\left(\frac{a}{2(s-a)}\right)^2} \left|^{(\rho-2\nu+2r,\lambda),(\rho-2\nu+2r,\lambda),A*}_{B*,(\rho,\lambda),(1,1)(1-\nu,1)}\right.]$$

$$- \aleph^{0,1;m,n;1,0}_{1,0;p_i,q_i+1;0,2;\tau_i;r}[{}^{z(s-a)^{-\lambda}}_{-\left(\frac{a}{2(s-a)}\right)^2} \left|^{(\rho,\lambda),(\rho,\lambda),A*}_{B*,(\rho,\lambda),(1,1)(1-\nu,1)}\right.] \;] \qquad (3.2)$$

Again taking $f(x) = x^{\nu}$ in Theorem 2.2 yields after a little simplification:

$$\int_0^{\infty} (x-a)^{\nu} (x+s)^{-\rho} \aleph^{m,n}_{p_i,q_i;\tau_i;r}[z(x+s)^{-\lambda} \left|^{A*}_{B*}\right.] dx$$

$$= \frac{\Gamma(\nu)}{(s+a)^{\rho-\nu-1}} \aleph^{m,n+1}_{p_i+1,q_i+1;\tau_i;r}[z(s+a)^{\lambda} \left|^{(-1+\rho-\nu,\lambda),A*}_{B*,(\rho,\lambda)}\right.] \qquad (3.3)$$

Provided that $\min\left\{\min\limits_{1\le j\le m}\operatorname{Re}\left(\rho-\nu-1+\lambda\tau_i\frac{b_{ji}}{\beta_{ji}}\right),\operatorname{Re}(\nu+1,s),\lambda\right\}>0.$

Similarly, if we take $f(x)=(1+a/x)^{k/2}\,p_n^k(1+2x/a)$ where $P_n^k(x)$ is the Legendre function ([3],p.1009,eqn(8.771(1)), in theorem 2.3, simply using ([2],p.216,eq.(16);p.294,eqn(5)), we have an interesting integral:

$$\int_0^\infty (1+a/x)^{k/2}P_n^k(1+a/x)(x+a)^{-\rho}\aleph^{m,n}_{p_i,q_i:\tau_i:r}[z(x+s)^{-\lambda}\Big|^{A^*}_{B^*}]dx$$

$$=\frac{a^{n+1}}{s^{\rho+n}}\sum_{r=0}^{\infty}\left(\frac{s-a}{s}\right)^r\frac{(n+1-k)_r}{r!}\aleph^{m+2,n}_{p_i+2,q_i+2:\tau_i:r}[z(s)^{\lambda}\Big|^{A^*,(1-\rho+k-r,\lambda),(1-\rho,\lambda)}_{(1-\rho-n-r,\lambda),(-\rho+n,\lambda),B^*}] \quad (3.4)$$

Provided that

$$\operatorname{Re}(k)<1,\ \max_{1\le j\le n}\operatorname{Re}\left(\lambda\tau_i\frac{1-a_{ji}}{\alpha_{ji}}-\rho+n\right)<0,\ \min\{\operatorname{Re}(s),\lambda\}>0,|\arg(a)|>0$$

Next, taking $f(x)=x^{\nu}$ in Theorem 2.4, a little simplification yields the following integral:

$$\int_0^\infty (x+a)^{-\nu-1}(x)^{-\rho}\aleph^{m,n}_{p_i,q_i+1:\tau_i:r}[z(x)^{\lambda}\Big|^{A^*}_{B^*,(\rho,\lambda)}]dx$$

$$=\frac{\Gamma(\nu)}{(s)^{-\rho+\nu+1}}\aleph^{m+1,n}_{p_i,q\ +1:\tau_i:r}[z(s+a)^{\lambda}\Big|^{A^*}_{(1-\nu+\rho,\lambda),B^*}] \quad (3.5)$$

$$\max_{1\le j\le n}\operatorname{Re}\left(\lambda\tau_i\frac{1-a_{ji}}{\alpha_{ji}}+\rho-\nu\right)<0,\ \lambda>0\quad,\quad \min_{1\le j\le m}\operatorname{Re}\left(\lambda\tau_i\frac{b_{ji}}{\beta_{ji}}+\rho,s\right)>0$$

Also, in Theorem 2.3, if we take $f(x)=x^{\eta-1}\aleph^{m,n}_{p_i,q_i:\tau_i:r}[zx^{\lambda}]$, and reduce the $\aleph^{m,n}_{p_i+1,q_i:\tau_i:r}$ involved in (2.8) to $\aleph^{m,0}_{p_i,q_i:\tau_i:r}$, we get a known result ([3],p.34), after a little simplification.

Again, if we take $\lambda=1,\rho=\beta,\tau_i=1,r=1$ and $\aleph^{m,n}_{p_i,q_i:\tau_i:r}$ occurring in (2.9) as

$\aleph^{2,0}_{1,2:\tau_i:r}\left[z(x+s)\Big|^{(1-\alpha,1)}_{(1-\gamma,1),(1-\delta,1)}\right]$,

We shall easily arrive at a result by Jain ([4], p.192) after a little simplification.

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